

Dartmouth College

## Dartmouth Digital Commons

---

Open Dartmouth: Published works by  
Dartmouth faculty

Faculty Work

---

10-10-2000

### Inhomogeneous Nucleation In a Quark-Hadron pPhase Transition

P. Shukla

*Bhabha Atomic Research Centre*

A. K. Mohanty

*Bhabha Atomic Research Centre*

S. K. Gupta

*Bhabha Atomic Research Centre*

Marcelo Gleiser

*Dartmouth College*

Follow this and additional works at: <https://digitalcommons.dartmouth.edu/facoa>



Part of the [Nuclear Commons](#)

---

#### Dartmouth Digital Commons Citation

Shukla, P.; Mohanty, A. K.; Gupta, S. K.; and Gleiser, Marcelo, "Inhomogeneous Nucleation In a Quark-Hadron pPhase Transition" (2000). *Open Dartmouth: Published works by Dartmouth faculty*. 1948.  
<https://digitalcommons.dartmouth.edu/facoa/1948>

This Article is brought to you for free and open access by the Faculty Work at Dartmouth Digital Commons. It has been accepted for inclusion in Open Dartmouth: Published works by Dartmouth faculty by an authorized administrator of Dartmouth Digital Commons. For more information, please contact [dartmouthdigitalcommons@groups.dartmouth.edu](mailto:dartmouthdigitalcommons@groups.dartmouth.edu).

# Inhomogeneous nucleation in a quark-hadron phase transition

P. Shukla, A. K. Mohanty, and S. K. Gupta

*Nuclear Physics Division, Bhabha Atomic Research Centre, Trombay, Mumbai 400 085, India*

Marcelo Gleiser

*Department of Physics and Astronomy, Dartmouth College, Hanover, New Hampshire 03755*

(Received 8 June 2000; published 10 October 2000)

The effect of subcritical hadron bubbles on a first-order quark-hadron phase transition is studied. These subcritical hadron bubbles are created due to thermal fluctuations, and can introduce a finite amount of phase mixing (quark phase mixed with hadron phase) even at and above the critical temperature. For reasonable choices of surface tension and correlation length, as obtained from the lattice QCD calculations, we show that the amount of phase mixing at the critical temperature remains below the percolation threshold. Thus, as the system cools below the critical temperature, the transition proceeds through the nucleation of critical-size hadron bubbles from a metastable quark-gluon phase (QGP), within an inhomogeneous background populated by an equilibrium distribution of subcritical hadron bubbles. The inhomogeneity of the medium results in a substantial reduction of the nucleation barrier for critical bubbles. Using the corrected nucleation barrier, we estimate the amount of supercooling for different parameters controlling the phase transition, and briefly discuss its implications to cosmology and heavy-ion collisions.

PACS number(s): 12.38.Mh, 64.60.Qb, 05.70.Fh, 25.75.-q

## I. INTRODUCTION

The hadronization of quark gluon plasma (QGP) possibly produced in the early universe or expected to be formed in relativistic heavy-ion collisions has been the focus of much attention during the past few years. However, the mechanism of hadronization (QCD phase transition) remains an open question. The prediction of lattice QCD on the order of the transition is still unclear, if physical masses for quarks are used [1]. Quenched QCD (no dynamical quarks) shows a first-order phase transition, albeit a weak one, with small surface tension and latent heat [2]. Assuming the transition to be first-order, homogeneous nucleation theory [3,4] has been invoked extensively to study the dynamics of the quark hadron phase transition both in the context of early universe as well as for the plasma produced during relativistic heavy-ion collisions [5–12]. In this picture, the transition is initiated by the nucleation of critical-size hadron bubbles from a supercooled metastable QGP phase. These hadron bubbles can grow against surface tension, converting the QGP phase into the hadron phase as the temperature drops below the critical temperature,  $T_C$ . This is indeed the case for a sufficiently strong first order transition, where the assumption of a homogeneous background of QGP is justified at the time when the nucleation begins. However, for a weak enough transition, the QGP phase may not remain in a pure homogeneous state even at  $T=T_C$ , due to pretransitional phenomena. For temperatures much above  $T_C$ , matter is in a pure QGP phase with the effective potential exhibiting one minimum at  $\phi=0$ . Here  $\phi$  is an effective scalar order parameter generally used to model the effective potential describing the dynamics of a phase transition. As the plasma expands and cools to some temperature  $T_1$ , an inflection point is developed away from the origin which on further cooling separates into a maximum at  $\phi=\phi_m$  and a local minimum at  $\phi=\phi_h$ , corresponding to the hadron phase. At  $T=T_C$ , the

potential is degenerate with a barrier separating the two phases. “Pretransitional phenomena” refers to nonperturbative dynamical effects above  $T_C$  in the range  $T_C \leq T \leq T_1$ . Such phenomena are known to occur in several areas of condensed matter physics, as in the case of isotropic to nematic phase transition in liquid crystals [13], and are also expected in the cosmological electroweak phase transition leading to large phase mixing at  $T=T_C$  [14]. In such cases, the phase transition may proceed either through percolation [14,15] or, if the phase mixing is below the percolation threshold, by the nucleation of critical bubbles in the background of isolated hadronic domains, which grow as  $T$  drops below  $T_C$ . In either case, the kinetics is quite different from what is expected on the basis of homogeneous nucleation [16]. We will argue that, for a wide range of physical parameters, a large amount of thermal phase mixing at  $T=T_C$  is expected to occur during the quark-hadron phase transition in the early universe, as well for the plasma produced in heavy-ion collisions [17]. For high enough temperatures and low enough cooling, large-amplitude thermal fluctuations will populate the new minimum at  $\phi=\phi_h$  in the range  $T_C \leq T \leq T_1$ . Although these fluctuations which are in the form of subcritical hadron bubbles will shrink and finally disappear, there will always be some nonzero number density of hadron bubbles at a given temperature  $T$ . In this work, we study the equilibrium density distribution of subcritical hadron bubbles for a wide spectrum of very weak to very strong first order QCD phase transition, using the formalism developed in Refs. [18–20]. It is found that the density of subcritical hadron bubbles builds up faster as the transition becomes weaker, leading, in some cases, to complete phase mixing at  $T=T_C$ . Further, using reasonable values for the surface tension and correlation length as obtained from lattice QCD calculations, we find that (although large) the amount of phase mixing remains below the percolation threshold. Therefore, the quark-hadron phase transition will begin with

the nucleation of critical-size hadron bubbles from a supercooled and inhomogeneous background of quark-gluon plasma. Since the background contains subcritical hadron bubbles, the homogeneous theory of nucleation needs to be modified. In Ref. [16], an approximate method was suggested to incorporate this inhomogeneity by modeling subcritical bubbles as Gaussian fluctuations, resulting in a large reduction in the nucleation barrier. Here, we will study inhomogeneous nucleation in the framework of homogeneous theory, but with a reduced nucleation barrier that accounts for the inhomogeneity of the medium. Finally, we also briefly discuss possible implications of inhomogeneous nucleation to relativistic heavy-ion collisions and cosmology.

The paper is organized as follows. In the next section, we begin with the discussion of a quartic double-well potential used to describe the dynamics of a first-order quark-hadron phase transition. The parameters of the potential are obtained in terms of relevant physical quantities such as critical temperature, surface tension and correlation length. In Sec. III, we estimate the equilibrium fraction of subcritical hadron bubbles from very weak to strong first-order phase transitions. We also estimate the reduction in the nucleation barrier by incorporating the presence of subcritical bubbles in the medium. Using this reduced barrier, we study nucleation and supercooling in Sec. IV. Finally, we present our conclusions in Sec. V.

## II. PARAMETRIZATION OF THE EFFECTIVE POTENTIAL

We consider a general form of the potential (or equivalently, the homogeneous part of the Helmholtz free energy density) to study the quark-hadron phase transition in terms of a real scalar order parameter  $\phi$  given by

$$V(\phi, T) = a(T)\phi^2 - bT\phi^3 + c\phi^4, \quad (1)$$

where  $b$  and  $c$  are positive constants. Ignatius *et al.* [21] use this parametrization to describe the phase transition from a QGP (symmetric phase) to a hadron phase (broken symmetry). The meaning of  $\phi$  is obvious for a symmetry-breaking transition, but the same description can be used if no symmetry is involved. The order parameter could then be related to energy or entropy density. The parameters  $a$ ,  $b$ , and  $c$  are determined in terms of surface tension ( $\sigma$ ), correlation length ( $\xi$ ), and critical temperature ( $T_C$ ). The potential has two minima, one at  $\phi_q = 0$  and the other at  $\phi_h = (3bT + \sqrt{9b^2T^2 - 32ac})/8c$ , which in our case will represent quark and hadron phases, respectively. These phases are separated by a maximum defined by  $\phi_m = (3bT - \sqrt{9b^2T^2 - 32ac})/8c$ . At  $T = T_C$ ,

$$V(\phi_q, T_C) = V(\phi_h, T_C) = 0, \quad (2)$$

having the required degeneracy. The above condition yields

$$\begin{aligned} a(T_C) &= b^2T_C^2/4c, \quad \phi_h(T_C) = bT_C/2c, \text{ and } \phi_m(T_C) \\ &= bT_C/4c. \end{aligned} \quad (3)$$

Using these relations, the barrier height at  $T_C$  can be written as

$$V_b = b^4T_C^4/256c^3. \quad (4)$$

Therefore, if the parameter  $c$  is kept fixed,  $b$  can be varied to characterize a wide spectrum of very weak to very strong first-order phase transitions. The transition is strong enough for large  $V_b$  and very weak or close to second order as  $V_b \rightarrow 0$ . In the following, we relate the parameters  $b$  and  $c$  to the surface tension and the correlation length in the quark phase. The surface tension can be defined as the one dimensional action given by

$$\sigma = \int dx \left[ \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + V(\phi) \right]. \quad (5)$$

Under the thin-wall limit,  $\Delta = |V(0) - V(\phi_h)| \rightarrow 0$ , the surface tension can be expressed as [22]

$$\begin{aligned} \sigma &= \int_0^{\phi_h} d\phi \sqrt{2V(\phi)}, \\ &= \frac{\sqrt{2}}{48} \frac{b^3T_C^3}{c^{5/2}}. \end{aligned} \quad (6)$$

Similarly, the correlation length around the quark phase is obtained using  $\xi_q = 1/\sqrt{V''(\phi)}|_{\phi=0} = 1/\sqrt{2a(T)}$ . At the critical temperature, using Eq. (3), we get

$$\xi_q(T_C) = \frac{\sqrt{2c}}{bT_C}. \quad (7)$$

From Eqs. (6) and (7) we get

$$c = \frac{1}{12\xi_q^3\sigma}, \quad b^2 = \frac{1}{6\xi_q^5\sigma T_C^2}, \quad (8)$$

in terms of the values of  $\sigma$  and  $\xi_q$  at  $T_C$ . The barrier height  $V_b$  can now be written as

$$V_b = \frac{3}{16} \frac{\sigma}{\xi_q(T_C)}. \quad (9)$$

Thus, the barrier height is proportional to the ratio  $\sigma/\xi_q$ . The transition becomes very weak as  $\sigma$  decreases and  $\xi_q$  increases. Here, we fix  $\xi_q = 0.5$  fm at  $T = T_C$  and vary  $\sigma$  to investigate phase transitions with different strengths. The temperature dependence of  $a$  is deduced by equating the depth of the second minimum with the pressure difference  $\Delta P$  between the two phases at all temperatures. This yields an equation

$$\Delta P = p_h - p_q = V(0) - V(\phi_h) = -(a(T) - bT\phi_h + c\phi_h^2)\phi_h^2, \quad (10)$$

which is solved to get the parameter  $a(T)$ , giving the temperature dependence of  $\xi_q$ . The surface tension will also have small temperature dependence which we ignore, as we are not going too far from the critical temperature. Thus, we

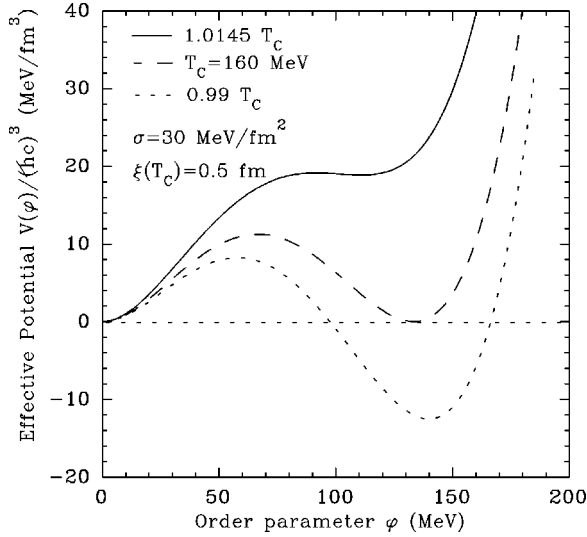


FIG. 1. The effective potential as a function of order parameter  $\phi$  at, below, and above  $T_c$ .

have parametrized the free-energy density in terms of the surface tension, correlation length, critical temperature and equation of state, which can be obtained from lattice QCD calculations. The bag equation of state which is a good depiction of the lattice results is used to calculate the quark/hadron pressure  $p_{q/h}$  as follows:

$$p_q = a_q T^4 - B, \quad p_h = a_h T^4, \quad (11)$$

where  $B = (a_q - a_h)T_c^4$  is the bag constant. The quark phase is assumed to consist of a massless gas of  $u$  and  $d$  quarks and gluons, while the hadron phase contains massless pions. Thus, the coefficients  $a_q$  and  $a_h$  are given by  $a_q = 37\pi^2/90$  and  $a_h = 3\pi^2/90$ . The critical temperature is taken as  $T_c = 160$  MeV.

Figure 1 shows the plot of  $V(\phi)$  as a function of  $\phi$  at three different temperatures for a typical value of  $\sigma = 30$  MeV/fm<sup>2</sup> and  $\xi_q(T_c) = 0.5$  fm. At  $T = T_c$ , the potential is degenerate with a large barrier that separates the two phases. Below  $T_c$ , the phase  $\phi = \phi_h$  has lower free-energy density, and the QGP phase becomes metastable. Above  $T_c$ , the potential has a metastable minima at  $\phi = \phi_h$  (hadron phase) as long as  $T$  remains below  $T_1$ . The temperature  $T_1$  [at which  $\phi_h = \phi_m$  and  $9b^2T_1^2 = 32a(T_1)c$ ] can be obtained analytically by solving Eq. (10) as

$$T_1 = \left[ \frac{B}{\frac{27}{B - \frac{16}{16}V_b}} \right]^{1/4} T_c. \quad (12)$$

It may be mentioned here that the dynamics of the phase transition has also been studied in Ref. [17] using a different form of the potential which has been parametrized as a fourth order polynomial in the energy density [5]. This form is unsuitable over a wide range of temperatures due to the persistence of metastability at much above and below  $T_c$ .

### III. MODEL FOR LARGE-AMPLITUDE FLUCTUATIONS

We closely follow the work of Refs. [16,19,20] to estimate the equilibrium density distribution of subcritical hadron bubbles by modeling them as Gaussian fluctuations with amplitude  $\phi_A$  and radius  $R$

$$\phi_{q \rightarrow h}(r) = \phi_A e^{-r^2/R^2} \quad \text{and} \quad \phi_{h \rightarrow q}(r) = \phi_A (1 - e^{-r^2/R^2}). \quad (13)$$

The amplitude  $\phi_A$  is the value of the field at the bubble's core away from the quark phase. For smooth interpolation between the two phases in the system,  $\phi_A \geq \phi_m$ . The free energy of a given configuration can then be found by using the general formula [22]

$$F = \int d^3r \left[ \frac{1}{2} (\nabla \phi(r))^2 + V(\phi(r)) \right]. \quad (14)$$

Using Eqs. (13) and (1) in Eq. (14) we get

$$F_{q \rightarrow h} \equiv F_h = \alpha_h R + \beta_h R^3 \quad \text{and} \quad F_{h \rightarrow q} \equiv F_q = \alpha_q R + \beta_q R^3, \quad (15)$$

where  $\alpha_h$ ,  $\beta_h$ ,  $\alpha_q$ , and  $\beta_q$  are given by

$$\alpha_h = \alpha_q = \frac{3\sqrt{2}}{8} \pi^{3/2} \phi_A^2, \quad \beta_h = \left[ \frac{\sqrt{2}a}{4} - \frac{\sqrt{3}bT}{9} \phi_A + \frac{c}{8} \phi_A^2 \right] \pi^{3/2} \phi_A^2, \quad (16)$$

and

$$\beta_q = \left( \frac{\sqrt{2}}{4} - 2 \right) a \pi^{3/2} \phi_A^2 - \left( -\frac{\sqrt{3}}{9} - 3 + \frac{3\sqrt{2}}{4} \right) b T \pi^{3/2} \phi_A^3 + \left( \frac{1}{8} + \frac{3\sqrt{2}}{2} - \frac{4\sqrt{3}}{9} - 4 \right) c \pi^{3/2} \phi_A^4. \quad (17)$$

It may be mentioned here that  $\alpha_h (= \alpha_q)$  is positive and is much greater than  $\beta_{h(q)}$ . Therefore, the free energy grows linearly for small values of  $R$ . Further, hadron bubbles of all configurations will be subcritical as long as  $\beta_{h(q)}$  is positive. At  $T = T_c$ , both  $\beta_h$  and  $\beta_q$  are positive for all amplitudes. However, below  $T_c$ ,  $\beta_h$  may become negative for some values of  $\phi_A$ . For such configurations, the free energy has a maximum at  $R_m = \sqrt{\alpha_h/3\beta_h}$  and these bubbles are not strictly subcritical. The same is true for  $\beta_q$  above  $T_c$ . We thus restrict the amplitudes  $\phi_A$  to the range where  $\beta_{h(q)}$  is positive. If not exactly the same, the limits of integration  $\phi_{\min}$  and  $\phi_{\max}$  for  $\phi_A$  are found to be quite close to  $\phi_m$  and  $\phi_h$ , respectively.

#### A. Equilibrium fraction of subcritical bubbles

There will be fluctuations from quark to hadron phase and back. To obtain the number density  $n_A$  of subcritical bubbles, we define the distribution function  $f \equiv \partial^2 n_A / \partial R \partial \phi_A$  where  $f(R, \phi_A, t) dR d\phi_A$  is the number density of bubbles with ra-

dus between  $R$  and  $R+dR$  and amplitude between  $\phi_A$  and  $\phi_A+d\phi_A$  at time  $t$ . It satisfies the Boltzmann equation [16,20]

$$\frac{\partial f(R, \phi_A, t)}{\partial t} = -|v| \frac{\partial f}{\partial R} + (1 - \gamma)G_h - \gamma G_q. \quad (18)$$

The first term on the RHS is the shrinking term. Here,  $|v|$  is the shrinking velocity, which we assume to be given by the velocity of sound ( $=1/\sqrt{3}$ ) in a massless gas. The second term is the nucleation term where  $G$  is the Gibbs distribution function defined as  $\Gamma = \int dR d\phi G$ . Here  $\Gamma_h$  is the nucleation rate per unit volume of subcritical bubbles from the quark phase to the hadron phase. Similarly  $\Gamma_q$  is the corresponding rate from the hadron phase to the quark phase. The factor  $\gamma$  is defined as the fraction of volume in the hadron phase and is obtained by summing over subcritical bubbles of all amplitudes and radii within this phase. The Gibbs distribution function is defined as [19,22]

$$G_{h/q} = AT^4 e^{-F_{h/q}(R, \phi_A)/T}, \quad (19)$$

where  $A$  is of  $\mathcal{O} \sim 1$  [22].

If the equilibration time scale is smaller than the expansion time scale of the system, we can obtain the equilibrium number density of subcritical bubbles by solving Eq. (18) with  $\partial f/\partial t = 0$ . Since the early universe expands at a much slower rate [10,11], the above assumption is quite reasonable in the context of the cosmological QCD phase transition. However, QGP produced during heavy ion collision may expand at a faster rate as compared to the early universe. In this case, it is possible that the density distribution of the subcritical bubbles will not attain full equilibrium. For simplicity, we will assume an equilibrium situation so that the present results on the fraction of subcritical bubbles and phase mixing can be considered as an upper limit. Using the boundary condition  $f(R \rightarrow \infty) = 0$ , we get the equilibrium distribution given by

$$f(R, \phi_A, T) = (1 - \gamma)W_S(R, \phi_A, T) - \gamma W_T(R, \phi_A, T), \quad (20)$$

where

$$W_S(R, \phi_A, T) = (A/|v|)T^4 \int_R^\infty e^{-(\alpha_h R' + \beta_h R'^3)/T} dR',$$

$$W_T(R, \phi_A, T) = (A/|v|)T^4 \int_R^\infty e^{-(\alpha_q R' + \beta_q R'^3)/T} dR'. \quad (21)$$

The equilibrium fraction  $\gamma$  of volume occupied by subcritical bubbles is given by

$$\gamma(\phi_{\min}, \phi_{\max}, R_{\min}, R_{\max}) = \int_{\phi_{\min}}^{\phi_{\max}} \int_{R_{\min}}^{R_{\max}} \frac{4\pi}{3} R^3 f(R, \phi_A, T) dR d\phi_A, \quad (22)$$

which is solved to get

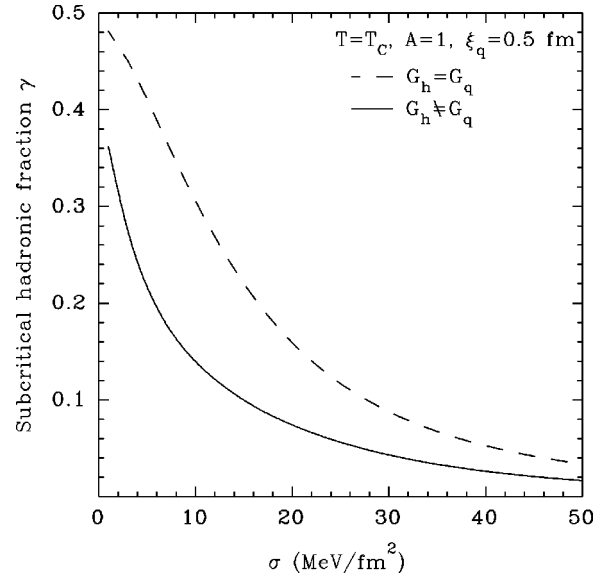


FIG. 2. Subcritical hadronic fraction  $\gamma$  as a function of surface tension  $\sigma$ .

$$\gamma = \frac{I_S}{1 + I_S + I_T}, \quad (23)$$

where

$$I_{S(T)} = \int_{\phi_{\min}}^{\phi_{\max}} \int_{R_{\min}}^{R_{\max}} \frac{4\pi}{3} R^3 W_{S(T)}(R, \phi_A, T) dR d\phi_A. \quad (24)$$

Here,  $\phi_{\min}$  and  $\phi_{\max}$  define the range within which both  $\beta_h$  and  $\beta_q$  are positive.  $R_{\min}$  is the smallest radius of the subcritical bubbles taken as  $\xi_q$ , the correlation length of the fluctuations. The  $R$  integration should be carried out over all bubbles with radii from  $R_{\min} = \xi_q$  to  $R_{\max} = \infty$ . For very weak transitions, both  $\alpha$  and  $\beta$  are very small and the  $R$  integration may not have good convergence. However, we found that the  $\gamma$  value is maximized when  $R_{\max}$  is about 3 to 4 fm. Therefore, we use  $R_{\max} = 3.5$  fm. This is a reasonable choice as bubbles with  $R \sim \xi_q$  will be statistically dominant and larger fluctuations have larger free energy and are exponentially suppressed.

Figure 2 shows the plot of the subcritical hadron fraction  $\gamma$  as a function of  $\sigma$  at  $T = T_c$  and at a fixed value of  $\xi_q(T_c) = 0.5$  fm. The fraction  $\gamma$  has been estimated (dashed curve) assuming that, for a degenerate potential,  $G_h \simeq G_q$ , as in Ref. [16]. This assumption is valid only for the configuration for which  $\phi_A = \phi_h$ . However, when we include other configurations in the range  $\phi_{\min}$  to  $\phi_{\max}$ , the integral  $I_T$  turns out to be always higher than  $I_S$  at  $T_c$ . Therefore,  $\gamma$  obtained using  $G_h \neq G_q$  is always lower than when the approximation  $G_h = G_q$  is used. In both cases, the value of  $\gamma$  increases with decreasing  $\sigma$ , i.e., when the transition becomes weak. It may be mentioned here that as per lattice QCD calculations without dynamical quarks [2],  $\sigma$  lies between 2 MeV/fm<sup>2</sup> and 10 MeV/fm<sup>2</sup>. There would be 15% to 30% phase mixing corresponding to these  $\sigma$  values, which is still below the



percolation threshold ( $\gamma \leq 0.3$ ). If  $\gamma > 0.3$ , the two phases will mix completely, the mean-field approximation for the potential breaks down, and the phase transition may proceed through percolation [15,20]. However, for a surface tension in the range  $2 \text{ MeV/fm}^2 \leq \sigma \leq 10 \text{ MeV/fm}^2$ , the phase transition will proceed through the formation of critical-size hadron bubbles from a supercooled metastable QGP phase. Since the QGP phase is no longer homogeneous, the dynamics of the phase transition will be quite different from what is expected on the basis of homogeneous nucleation theory [16]. We refer to it as “inhomogeneous nucleation.”

We would also like to mention here that the present results are in disagreement with the findings of Ref. [17], where a large fraction of subcritical hadron phase was found at and above  $T_C$ . This scenario is highly unrealistic and probably could be due to the choice of the potential parametrization, which shows a metastable hadron phase much above  $T_C$ . Therefore, the authors of Ref. [17] found a finite fraction of hadron phase at temperatures as high as twice  $T_C$ . Furthermore, the value of  $\gamma$  strongly depends on how the shrinking term is incorporated in the calculation. In our case, it is proportional to the gradient ( $\partial f / \partial R$ ) that appears in the kinetic equation (18) in a natural way, whereas in Ref. [17], a specific assumption is made to take into account the shrinking of the hadronic volume.

### B. The total free energy of subcritical bubbles and the nucleation barrier

The nucleation rate in the standard theory [3,4] which neglects phase mixing, is given by

$$I \approx AT^4 e^{-F_C/T}. \quad (25)$$

Here  $F_C$  is free energy needed to form a critical bubble in the homogeneous metastable background. For an arbitrary thin-wall spherical bubble of radius  $R$  and amplitude  $\phi_{\text{thin}} \leq \phi_h$ , the free energy of the bubble takes the well-known form

$$F_{\text{thin}}(R) = -\frac{4\pi}{3}R^3\Delta V + 4\pi R^2\sigma. \quad (26)$$

In the above,  $\Delta V$  is defined as the difference in free-energy density between the background medium and the bubble's interior. For a homogeneous background (metastable), we can write

$$\Delta V \equiv \Delta V_0 = V(0) - V(\phi_h). \quad (27)$$

If there is significant phase mixing in the background metastable state, its free energy is no longer  $V(0)$ . One must also account for the free energy density of the nonperturbative large amplitude fluctuations. Following Ref. [16], we write the free energy density of the metastable state as  $V(0) + \mathcal{F}_{\text{sc}}$ , where  $\mathcal{F}_{\text{sc}}$  is the extra free energy density which can be estimated from the density distribution of subcritical bubbles as follows:

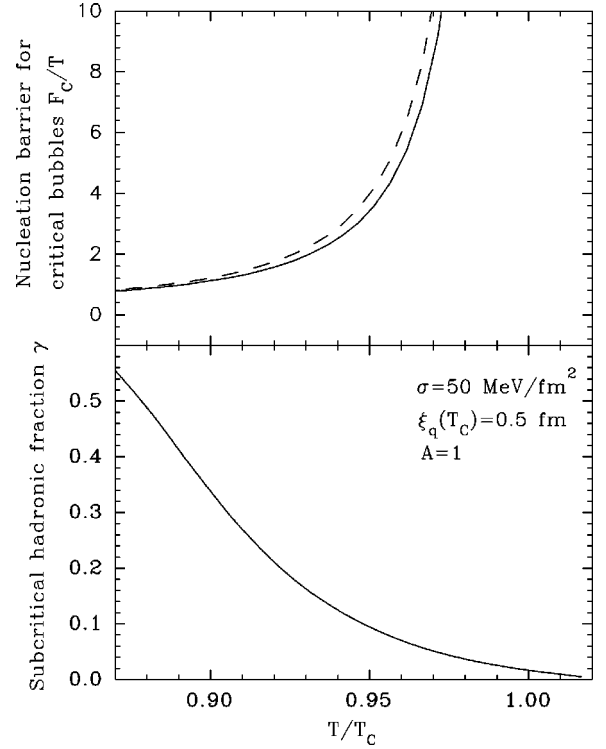


FIG. 3. The nucleation barrier  $F_C/T$  for critical bubbles with (solid line) and without subcritical bubble correction (dashed curve) as function of temperature for  $\sigma = 50 \text{ MeV/fm}^2$  is shown in the upper panel. Corresponding subcritical hadron fraction  $\gamma$  is shown in the lower panel.

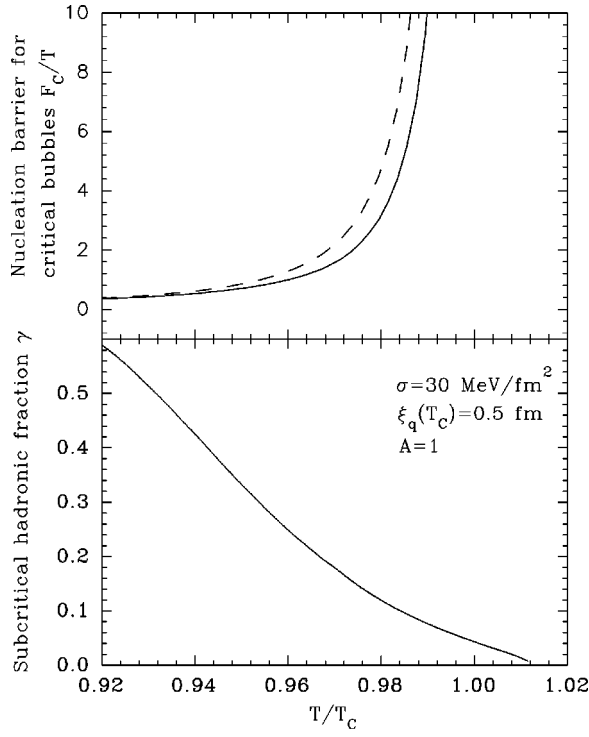
$$\begin{aligned} \mathcal{F}_{\text{sc}} &\approx \int_{\phi_{\min}}^{\phi_{\max}} \int_{R_{\min}}^{R_{\max}} F_h(R, \phi_A, T) f(R, \phi_A, T) dR d\phi_A, \\ &= (1 - \gamma) \int_{\phi_{\min}}^{\phi_{\max}} \int_{R_{\min}}^{R_{\max}} F_h W_S dR d\phi_A \\ &\quad - \gamma \int_{\phi_{\min}}^{\phi_{\max}} \int_{R_{\min}}^{R_{\max}} F_h W_T dR d\phi_A. \end{aligned} \quad (28)$$

Once we know the hadronic fraction  $\gamma$  and the free energy  $F_h$  for a bubble of a given radius  $R$  and amplitude  $\phi_A$ , we can estimate the free-energy density correction due to the presence of Gaussian subcritical bubbles.

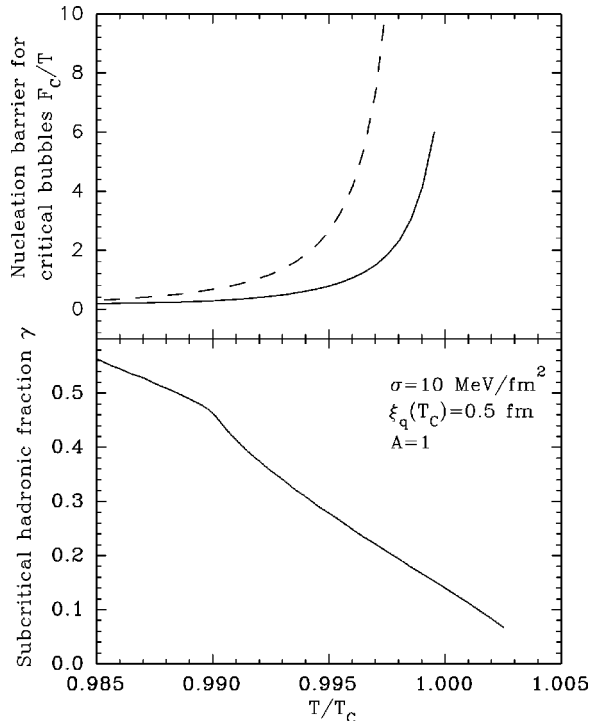
Since, for a critical size bubble,  $\partial F / \partial R|_{R_C} = 0$ , we can use Eq. (26) to obtain the free energy needed to form a thin-wall critical bubble in a background of subcritical bubbles,

$$F_C = \frac{4\pi}{3} \sigma R_C^2, \quad R_C = \frac{2\sigma}{\Delta V_0 + \mathcal{F}_{\text{sc}}}. \quad (29)$$

For a very strong first-order phase transition, the subcritical bubbles are suppressed ( $\mathcal{F}_{\text{sc}} \rightarrow 0$ ), and both  $F_C$  and  $R_C$  approach the homogeneous background expression. However, in the presence of subcritical bubbles, extra free energy becomes available in the medium, reducing the nucleation barrier. In other words, the extra background energy enhances the nucleation of critical bubbles. To illustrate this, we have plotted  $F_C/T$  and  $\gamma$  as a function of  $T/T_C$  in Figs. 3–5 with

FIG. 4. Same as Fig. 3 but at  $\sigma = 30$  MeV/fm<sup>2</sup>.

$\sigma$  values of 50 MeV/fm<sup>2</sup>, 30 MeV/fm<sup>2</sup>, and 10 MeV/fm<sup>2</sup>, respectively, which are widely used in the literature. As evident, with decreasing temperature, the nucleation barrier decreases and the subcritical hadron fraction  $\gamma$  increases. The reduction in barrier height due to  $\mathcal{F}_{sc}$  (or due to  $\gamma$ ) is more significant for lower values of  $\sigma$ , corresponding to a weaker

FIG. 5. Same as Fig. 3 but at  $\sigma = 10$  MeV/fm<sup>2</sup>.

transition. Since the height of the nucleation barrier decreases, the nucleation rate will also be enhanced, reducing the amount of supercooling further. The time evolution of the temperature and the supercooling are discussed in the next section.

#### IV. NUCLEATION AND SUPERCOOLING

As mentioned before, the background metastable state is inhomogeneous due to subcritical hadron bubbles. It is now possible to study the kinetics of the nucleation of critical hadron bubbles using the corrected nucleation rate, as obtained in the previous section. In the present work, the prefactor in the nucleation rate is taken as  $AT^4$  [see Eq. (25)]. In our previous work, [9], we have used a prefactor derived by Csernai and Kapusta [6] for a dissipative QGP. In Ref. [23], Ruggeri and Friedman had derived a prefactor for a non-dissipative QGP. Recently, using a more general formalism, we have also derived a prefactor [24] which has both dissipative and nondissipative components corresponding to Refs. [6] and [23], respectively. However, for consistency with the subcritical formalism, we use a more generic form  $I_0 = AT^4$ , with  $A$  a constant of order unity, as used in many studies of quark-hadron phase transition (see, for example, Refs. [10,11]). The question of how to estimate the prefactor appearing in the nucleation rate of subcritical bubbles remains open. Using the nucleation rate  $I(T)$ , the fraction  $h$  of space which has been converted to hadron phase due to nucleation of critical bubbles and their growth can be calculated. If the system cools to  $T_c$  at a proper time  $\tau_c$ , then at some later time  $\tau$  the fraction  $h$  is given by [6],

$$h(\tau) = \int_{\tau_c}^{\tau} d\tau' I(T(\tau')) [1 - h(\tau')] V(\tau', \tau). \quad (30)$$

Here,  $V(\tau', \tau)$  is the volume of a critical bubble at time  $\tau$  which was nucleated at an earlier time  $\tau'$ ; this takes into account the bubble growth. The factor  $[1 - h(\tau)]$  accounts for the available space for new bubbles to nucleate. The model for bubble growth is simply taken as [25]

$$V(\tau', \tau) = \frac{4\pi}{3} \left( R_c(T(\tau')) + \int_{\tau'}^{\tau} d\tau'' v(T(\tau'')) \right)^3, \quad (31)$$

where  $v(T) = 3[1 - T/T_c]^{3/2}$  is the velocity of the bubble growth at temperature  $T$  [6,26]. The evolution of the energy density in 1+1 dimensions is given by

$$\frac{de}{d\tau} + \frac{\omega}{\tau} = 0. \quad (32)$$

The energy density  $e$ , enthalpy density  $\omega$  and the pressure  $p$  in pure QGP and hadron phases are given by the bag model equation of state. In the transition region, the  $e$  and  $\omega$  at a time  $\tau$  can be written in terms of the hadronic fraction as

$$e(\tau) = e_q(T) + [e_h(T) - e_q(T)]h(\tau),$$

$$\omega(\tau) = \omega_q(T) + [\omega_h(T) - \omega_q(T)]h(\tau). \quad (33)$$

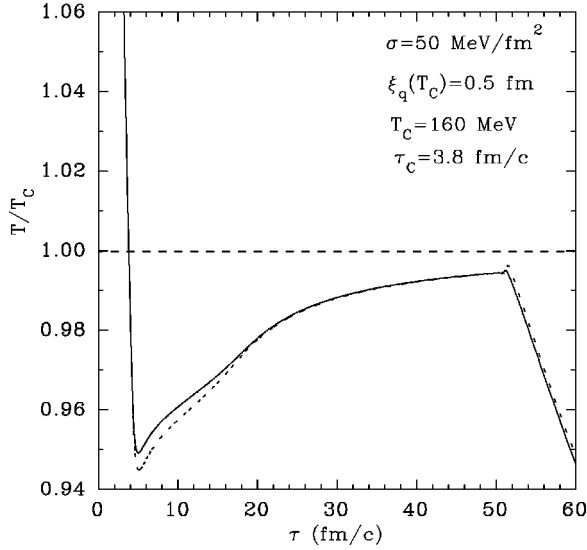


FIG. 6. The temperature variation as a function of proper time with (solid curve) and without subcritical bubble correction (dashed curve) for  $\sigma = 50$  MeV/fm<sup>2</sup>.

Equations (30), (32), and (33) are solved to get the temperature as a function of time in the mixed phase [9] with the initial conditions for temperature  $T_0 = 250$  MeV and proper time  $\tau_0 = 1$  fm/c at  $T_c = 160$  MeV. After getting  $T$  and  $h$  as a function of time, the density of nucleating bubbles at a time  $\tau$  can be obtained in our model as

$$N(\tau) = \int_{\tau_c}^{\tau} d\tau' I(T(\tau')) [1 - h(\tau')]. \quad (34)$$

The density  $N$  would increase as the temperature drops below  $T_c$  and would ultimately saturate as  $h$  increases.

Figure 6 shows the temperature variation as a function of proper time  $\tau$  at  $\sigma = 50$  MeV/fm<sup>2</sup>. As the system cools below  $T_c$ , the nucleation barrier decreases and also  $\gamma$  increases. If only homogeneous nucleation (dashed curve) is considered, the system will supercool up to  $0.945 T_c$ . At this temperature, the hadronic fraction  $\gamma$  has reached 10% (see Fig. 3), which corrects the amount of supercooling (solid curve) by about  $\sim 10\%$  (up to  $0.95 T_c$ ). Figure 7 shows a similar study at  $\sigma = 30$  MeV/fm<sup>2</sup>. Since the nucleation barrier reduces with decreasing  $\sigma$ , the system supercools only up to  $0.98 T_c$  under homogeneous nucleation. The hadronic fraction  $\gamma$  corresponding to this value is  $\sim 12 - 13\%$  (See Fig. 4) which reduces the amount of supercooling by about  $\sim 20\%$  (up to  $0.984 T_c$ ). For  $\sigma$  around 10 MeV/fm<sup>2</sup>, the supercooling will be reduced further (up to  $\sim 0.997 T_c$ ). Lattice QCD calculations predict a surface tension even smaller than 10 MeV/fm<sup>2</sup>, indicating a very weak first order transition. Although supercooling will be reduced further with decreasing  $\sigma$ , we do not use very small  $\sigma$  due to increased numerical inaccuracy. Further, it may be mentioned here that, although the fraction  $\gamma$  grows with decreasing  $\sigma$ , we never encountered  $\gamma$  greater than 0.3: we remained within the subpercolation regime throughout our analysis.

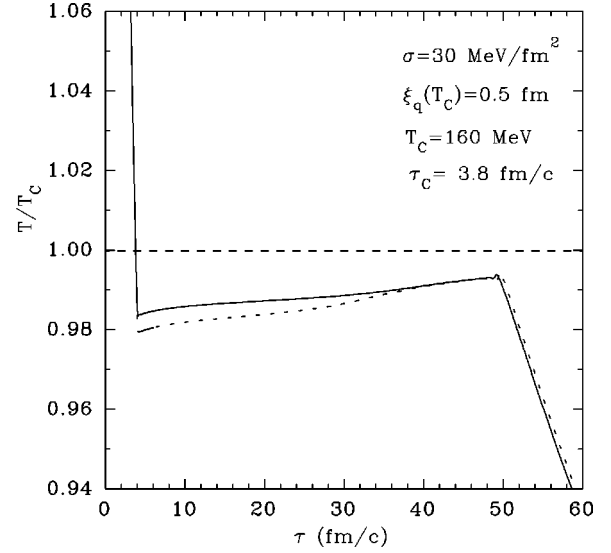


FIG. 7. Same as Fig. 6 but at  $\sigma = 30$  MeV/fm<sup>2</sup>.

Apart from  $\sigma$ , the amount of supercooling also depends on  $\tau_c$ , the time taken by the system to cool from  $T_0$  to  $T_c$ . In QGP phase, the solution of Eq. (32) ( $T^3\tau = \text{const}$ ) predicts  $\tau_c = \tau_0(T_0/T_c)^3$ . The choice of  $\tau_0 = 1$  fm,  $T_0 = 250$  MeV and  $T_c = 160$  MeV results in  $\tau_c = 3.8$  fm/c. However, formation of QGP with higher initial temperature (as high as 3 to 4 times  $T_c$  resulting in large  $\tau_c$ ) cannot be ruled out at RHIC and LHC energies [27]. Therefore, we have also studied the effect of  $\tau_c$  on supercooling, specifically, on the hadronization rate as well as on the density of nucleating bubbles. Figures 8(a) and 8(b) show the plots of  $N(\tau)$  and  $h(\tau)$  as a function of  $\tau$  for two typical values of  $\tau_c$  (3.8 fm/c and 25 fm/c) corresponding to  $\sigma$

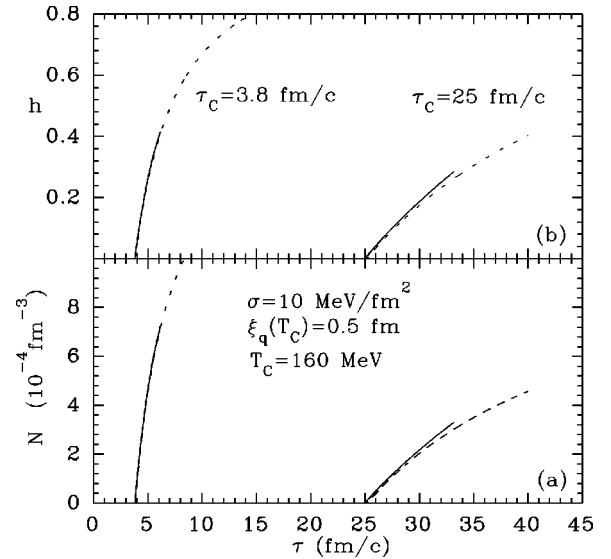


FIG. 8. (a) Density of nucleating bubbles as a function of proper time with (solid curve) and without subcritical bubble correction (dashed curve) for  $\sigma = 10$  MeV/fm<sup>2</sup>. (b) The hadronic fraction  $\gamma$  as a function of  $\tau$ . The left curves are for  $\tau_c = 3.8$  fm/c and the right curves for  $\tau_c = 25$  fm/c.



$=10 \text{ MeV/fm}^2$  both with (solid curve) and without (dashed curve) inhomogeneity corrections. A general observation (both with and without correction) is that the amount of supercooling, the rates of hadronization and bubble nucleation are reduced when  $\tau_c$  becomes larger. Although supercooling reduces with increasing  $\tau_c$ , the system will get reheated at a higher temperature and also will encounter a larger nucleation barrier as compared to the case when  $\tau_c$  is small. As a result, the rate of hadronization and also the rate of increase of density of the nucleating hadron bubbles will proceed at a slower rate when  $\tau_c$  is large. However, the reverse happens when the inhomogeneity correction is applied. Even though the medium gets heated up earlier, the reduction in the barrier height is quite significant as  $T$  approaches  $T_C$ . Another parameter that affects both  $N(\tau)$  and  $h(\tau)$  is the expansion rate of the medium, i.e., the rate of change of temperature between  $\tau_c$  and  $\tau$ , which also depends on  $\tau_c$ . The overall effect is that both  $N(\tau)$  and  $h(\tau)$  rise faster as compared to their homogeneous counterparts (see Fig. 8 for  $\tau_c = 25 \text{ fm/c}$ ), particularly when  $\tau_c$  is very large. (Compare the left and right curves on Fig. 8.)

For weak enough transition, the presence of inhomogeneity may also affect several observables which can be detected experimentally. For example, the faster rate of hadronization at large  $\tau_c$  as compared to its homogeneous counterpart will lower the amount of entropy production, which, in turn, will affect the final hadron multiplicity distributions. Although not studied here, the bubble size distribution will also be affected by the dynamics of nucleation [8]. Since the nucleating bubble will act as a source of pion emission, the effect of inhomogeneity can also be inferred through interferometry measurements.

In a cosmological context, the value of  $\tau_c$  is much larger than what was quoted here. Since the presence of inhomogeneities weakens the transition, more critical bubbles will be nucleated per unit volume, decreasing the interbubble distance ( $d \approx N^{-1/3}$ ); the presence of subcritical bubbles can be thought as seeds for nucleation. As a consequence, the transition will produce smaller fluctuations in baryon number, protecting homogeneous nucleosynthesis. Although the present study is indicative enough of the reduction in the

mean interbubble separation as compared to homogeneous nucleation, a quantitative estimate would require a more detailed analysis, including expansion. However, since the cosmological expansion rate is typically much slower than the subcritical bubble nucleation rate, we believe our results for the interbubble distance will carry on in this case as well.

## V. CONCLUSIONS

We have estimated the amount of phase mixing due to subcritical hadron bubbles from very weak to very strong first-order phase transitions. With a reasonable set of values for the surface tension and correlation length (as obtained from lattice QCD calculations), we found that phase mixing is small at  $T=T_C$ , building up as the temperature drops further. We have shown that the system does not mix beyond the percolation threshold, allowing us to describe the dynamics of the phase transition on the basis of homogeneous nucleation theory with a reduced nucleation barrier. Accordingly, we have found an enhancement in the nucleation rate which further reduces the amount of supercooling. Although we have not included cosmological expansion in our analysis, we believe that our results indicate that the presence of an inhomogeneous background of subcritical bubbles will decrease the interbubble mean distance, and thus the fluctuations in baryon number which could damage homogeneous nucleosynthesis.

We have assumed that the equilibration time scale for subcritical fluctuations is much larger than the cooling time scale of the system. This may be the case for a quark-hadron phase transition in the early universe, where the expansion rate is quite slow. In the case of QGP produced at RHIC and LHC, the cooling rate is much faster than cosmological time-scales, and the subcritical bubbles density distribution may not attain full equilibration. We are presently investigating this issue in more detail. However, the present results should provide an upper bound on the fraction of subcritical hadron bubbles and their effect on supercooling and nucleation rates.

## ACKNOWLEDGMENTS

M.G. is partially supported by a National Science Foundation Grant No. PHYS-9453431.

- 
- [1] Y. Iwasaki, K. Kanaya, S. Kaga, S. Sakai, and T. Yoshie, in *Lattice 94*, Proceedings of the International Symposium, Germany, edited by F. Karsch *et al.* [Nucl. Phys. B (Proc. Suppl.) **42**, 499 (1995)].
  - [2] Y. Iwasaki, K. Kanaya, L. Karkkainen, K. Rummukainen, and T. Yoshie, Phys. Rev. D **49**, 3540 (1994); B. Beinlich, F. Karsch, and A. Piekert, Phys. Lett. B **390**, 268 (1997).
  - [3] J. S. Langer, Ann. Phys. (N.Y.) **41**, 108 (1967); **54**, 258 (1969).
  - [4] J. S. Langer and L. A. Turski, Phys. Rev. A **8**, 3230 (1973).
  - [5] L. P. Csernai and J. I. Kapusta, Phys. Rev. D **46**, 1379 (1992).
  - [6] L. P. Csernai and J. I. Kapusta, Phys. Rev. Lett. **69**, 737 (1992).
  - [7] E. E. Zabrodin, L. V. Bravina, L. P. Csernai, H. Stöcker, and W. Griener, Phys. Lett. B **423**, 373 (1998).
  - [8] E. E. Zabrodin, L. V. Bravina, H. Stöcker, and W. Griener, Phys. Rev. C **59**, 894 (1999).
  - [9] P. Shukla, S. K. Gupta, and A. K. Mohanty, Phys. Rev. C **59**, 914 (1999); **62**, 039901(E) (2000).
  - [10] G. M. Fuller, G. J. Mathews, and C. R. Alcock, Phys. Rev. D **37**, 1380 (1988).
  - [11] B. S. Meyer, C. R. Alcock, and G. J. Mathews, Phys. Rev. D **43**, 1079 (1991).
  - [12] J. Ignatius, K. Kajantie, H. Kurki-Suonio, and M. Laine, Phys. Rev. D **50**, 3738 (1994).
  - [13] T. Stinton III and J. Lister, Phys. Rev. Lett. **25**, 503 (1970); H.

- Zink and W. H. de Jeu, *Mol. Cryst. Liq. Cryst.* **124**, 287 (1985).
- [14] M. Gleiser and E. W. Kolb, *Phys. Rev. Lett.* **69**, 1304 (1992); M. Gleiser and M. Trodden, hep-ph/9911380.
- [15] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, Washington, DC, 1992).
- [16] M. Gleiser and A. F. Heckler, *Phys. Rev. Lett.* **76**, 180 (1996).
- [17] B. K. Agrawal and S. Digal, *Phys. Rev. D* **60**, 074007 (1999).
- [18] M. Gleiser, E. W. Kolb, and R. Watkins, *Nucl. Phys.* **B364**, 411 (1991).
- [19] G. Gelmini and M. Gleiser, *Nucl. Phys.* **B419**, 129 (1994).
- [20] M. Gleiser, A. F. Heckler, and E. W. Kolb, *Phys. Lett. B* **405**, 121 (1997).
- [21] J. Ignatius, K. Kajantie, H. Kurki-Suonio, and M. Laine, *Phys. Rev. D* **49**, 3854 (1994).
- [22] A. Linde, *Nucl. Phys.* **B216**, 421 (1983).
- [23] Franco Ruggeri and William A. Friedman, *Phys. Rev. D* **53**, 6543 (1996).
- [24] P. Shukla, A. K. Mohanty, and S. K. Gupta, hep-ph/0005219, *Phys. Rev. D* (to be published).
- [25] S. Weinberg, *Astrophys. J.* **168**, 175 (1971).
- [26] J. C. Miller and O. Pantano, *Phys. Rev. D* **40**, 1789 (1989); **42**, 3334 (1990).
- [27] E. Shuryak, *Phys. Rev. Lett.* **68**, 3270 (1992).